Sato-Tate distributions for the twists of the Fermat and Klein quartics

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The Sato-Tate Conjecture: motivation

Let us consider the (complex multiplication) elliptic curve given by the equation

$$E:\ y^2 = x^3 + x/\mathbb{Q}.$$

It has good reduction at every prime. In the table below, we show the number of points of this curve over the finite field $\mathbb{F}_p$ for the first values of the prime p.

p	3	5	7	11	13	17	19	23	29
$\#E(\mathbb{F}_p) = 1 + p - a_1(p)$	4	4	8	12	20	16	20	24	20

Here, $a_1(p)$ demotes the trace of the Frobenius endomorphism.

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If we denote the ℓ-Tate module by $V_\ell(E) := T_\ell(E) \otimes \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^2$ and we consider the natural Galois representation attached to it

$$\rho_\ell : \ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_\ell),$$

the image of a Frobenius element in $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfies the equation

$$L_p(E/\mathbb{Q}, T) := T^2 - a_1(p)T + p = 0.$$

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Weil's Conjecture implies that $a_1(p)$ is an integer number in the interval $[-2\sqrt{p}, 2\sqrt{p}]$. So, we can work with the normalized coefficient

$$\overline{a}_1(p) = a_1(p)/\sqrt{p} \in [-2, 2].$$

We can also define the number $\theta_p = \arccos(\frac{\overline{a}_1(p)}{2})$ $(2\cos(\theta_p) = \overline{a}_1(p))$.

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We can consider the functions $\overline{a}_1$ and θ as **random variables** from the set of primes of good reduction of the elliptic curve E to the intervals $[-2, 2]$ and $[0, \pi]$ respectively.

$$\overline{a}_1 : \{\text{primes of good reduction}\} \longrightarrow [-2, 2],$$

$$\theta : \{\text{primes of good reduction}\} \longrightarrow [0, \pi].$$

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Let E be an elliptic curve defined over a number field k with CM. Then the random variable $\overline{a}_1$ is equidistributed with respect to the traces of the matrices in the groups $\mathrm{U}(1)$ *or* $\mathrm{N}(\mathrm{U}(1))$ *(normalizer of* $\mathrm{U}(1)$ *in* $\mathrm{SU}(2)$*) depending on whenever the CM is also defined or not over k.*

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`https://math.mit.edu/~drew/g1_D2_a1f.gif`

The (original) Sato-Tate Conjecture

Conjecture (Sato-Tate, 60's)

Let E be an elliptic curve defined over a number field k without CM. The random variable $\overline{a}_1$ has the semicircular distribution. Equivalently, the angles θ are equidistributed with respect to the measure $\frac{2}{\pi}\sin^2\theta \, d\theta$ on $[0,\pi]$. Which is again equivalent to say, that the polynomials $L_{\mathfrak{p}}(E/k, T/\sqrt{\mathrm{N}\,\mathfrak{p}})$ are equidistributed with respect to the characteristic polynomials of the elements of the algebraic group USp_2

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★The Sato-Tate conjecture is proved (2006) for totally real number fields after some works of Clozel, Harris, Shepherd-Barron and Taylor. For CM fields there is the 2018 preprint by Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor, and Thorne.

The Generalize Sato-Tate Conjecture

Let A/k be an abelian variety defined over a number field k of dimension g. Given a prime ℓ of good reduction, we have the Galois representation $\rho_\ell : \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$. The image by this representation of a Frobenius $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{G}_k$ satisfies

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where again, Weil's Conjectures imply that $a_i \in \left[-(\mathrm{N}\,\mathfrak{p})^{i/2}\binom{2g}{i}, (\mathrm{N}\,\mathfrak{p})^{i/2}\binom{2g}{i}\right] \cap \mathbb{Z}$, and that the polynomial $L_\mathfrak{p}(A/k, T/\sqrt{\mathrm{N}\,\mathfrak{p}})) = \sum_{i=0}^{2g}(-1)^i \overline{a}_i(\mathfrak{p}) T^i$ is symmetric and symplectic. Which means that the numbers

$$\overline{a}_{2g-i}(\mathfrak{p}) = \overline{a}_i(\mathfrak{p}) = a_i q^{i/2} \in \left[-\binom{2g}{i}, \binom{2g}{i}\right]$$

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What's about the distribution of $\overline{a}_i$'s and, how to describe it?

The Generalize Sato-Tate Conjecture

We will use a group whose elements are matrices to conjecture that the distribution of the $\overline{a}_i$'s will be the same that the distribution of the coefficients of the characteristic polynomials of the matrices in such group. This group will be called the **Sato-Tate group**, and denoted by $\mathrm{ST}_k(A)$. It will be a compact Lie group and then, there will be a unique Haar measure up to rescaling.

For instance, for the genus 1 case, we got three different options

$$\mathrm{ST}_k(E) = \begin{cases} \mathrm{SU}(2) & \text{non CM case} \\ \mathrm{U}(1) & \text{CM defined over } k \\ \mathrm{N}(\mathrm{U}(1)) & \text{CM not defined over } k \end{cases}$$

The Generalize Sato-Tate Conjecture

More explicitly, we have:

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} :\ a, b \in \mathbb{C}, \text{ s.t. } a\overline{a} + b\overline{b} = 1 \right\}$$

$$\mathrm{U}(1) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} :\ \theta \in [0, 2\pi] \right\}$$

$$\mathrm{N}(\mathrm{U}(1)) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix} :\ \theta \in [0, 2\pi] \right\}$$

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Conjecture (Generalized Sato-Tate)

Let A/k be an abelian variety defined over a number field k. Then, the local factors $L_{\mathfrak{p}}(A/k, T/(\mathrm{N}\,\mathfrak{p})^{1/2})$ are equidistributed with respect to the distribution of the characteristic polynomials of the matrices in $\mathrm{ST}_k(A)$ with respect to the Haar measure μ.

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★There exists an extra generalization to this conjecture due to Jean-Pierre Serre. This generalization is for *motives*, but we can think in a simpler one for varieties, where we have to switch the Tate module $V_\ell(A)$ by the cohomology set $\mathrm{H}_{et}^{\dim(X)}(X, \mathbb{Q}_\ell)$.

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★Johansson proved that the conjecture is true for complex multiplication abelian varieties.

The Sato-Tate group

Let A/k be an abelian variety of dimension g defined over a number field. Let ℓ be a prime of good reduction. Consider the Galois representation:

$$\rho_\ell : \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{Aut}(\mathrm{V}_\ell(A)) = \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$$

Denote by $G_{k,\ell}$ the kernel in $\mathrm{Gal}(\overline{k}/k)$ of the cyclotomic character and by G_ℓ the Zariski closure of the image of $G_{k,\ell}$ by the representation ρ_ℓ, which we view as a $\mathbb{Q}_\ell$-algebraic subgroup of Sp_{2g} (Weil pairing allows us to consider only symmetric matrices). Choose an embedding $\iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and denote by $G_{\ell,\iota} \subseteq \mathrm{Sp}_{2g}(\mathbb{C})$ the group of complex points of $G_\ell \otimes_\iota \mathbb{C}$.

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Definition (Sato-Tate group)

The Sato-Tate group of the abelian variety A/k is a maximal compact subgroup of $G_{\ell,\iota}$. It is a compact Lie group, and we denote it by $\mathrm{ST}_k(A)$.

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★ The Sato-Tate group is well-defined: there is up to conjugation a unique compact subgroup and the definition does not depend on the choice of the prime ℓ and the embedding ι. On the other hand, it depends on the field of definition of the abelian variety, so it is sensitive to base field extension.

The Sato-Tate group

Serre states some axioms that a general Sato-Tate group may satisfy. These axioms are useful in order to list all the possible Sato-Tate groups of certain varieties of fixed dimension.

Conjecture (Serre)

The Sato-Tate group $\mathrm{ST}_k(A)$ *of an abelian variety of dimension* g *satisfies the following properties:*

i) *It is a closed subgroup of* $\mathrm{USp}_{2g}(\mathbb{C})$*.*

ii) *(Hodge condition) There exists a subgroup* H*, called a Hodge circle, which is the image of a homomorphism* $\theta : \mathrm{U}(1) \longrightarrow \mathrm{ST}_k^0(A)$ *with some extra properties.*

iii) *(Rationality condition) For each component* C *of* $\mathrm{ST}_k(A)$ *and for each character* χ *of* GL_{2g}*, the expected value*

$$\int_{g \in C} \chi(g)\mu(g)$$

is an integer.

Moments

Proposition ($g \leq 2$, Kedlaya-Sutherland)

Under the generalized Sato-Tate conjecture, the moments of the random variables $\overline{a}_i$ exist and determine the distribution. Moreover, the expected values $M_n(\overline{a}_i) = \mathrm{E}[\overline{a}_i^n]$ are integer numbers.

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Example

The moment sequence for the three genus 1 case are

$$\begin{aligned} \mathrm{SU}(2) &: 1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, \dots \\ \mathrm{U}(1) &: 1, 0, 2, 0, 6, 0, 20, 0, 70, 0, 252, 0, \dots \\ \mathrm{N}(\mathrm{U}(1)) &: 1, 0, 1, 0, 3, 0, 10, 0, 35, 0, 126, 0, \dots \end{aligned}$$

Genus 2: [KS], [FKRS] and [FS]

★ Fité, Kedalaya, Rotger and Sutherland prove, using Serre axioms that there are 55 Sato-Tate possibilities for the Sato-Tate group of an abelian surface.

★ They prove that in fact, only 52 of them appear as Sato-Tate groups, and even more, all of them are reached by jacobians of genus 2 curves. Among these 52 groups, only 34 of them can be realized with abelian varieties defined over $\mathbb{Q}$.

★ They find candidate curves that attain these Sato-Tate groups and provide numerically evidences of the matches.

★ After Johansson's results, the Sato-Tate conjecture is proven for the genus 2 curves with CM. For example, for the curves $y^2 = x^6 + 1$ and $y^2 = x^5 - x$. Fité and Sutherland compute the Sato-Tate groups for them and their twists, obtaining 18 out of the 52 possibilities for these groups. In order to compute these groups, they use the results of Banaszak and Kedlaya concerning the Lefschez group.

★ `https://math.mit.edu/~drew/st2/g2SatoTateDistributions.html`

Genus 3: some distributions

Theorem (Fité, L.G., Sutherland)

The following hold:

1. *There are* 54 *distinct Sato-Tate groups of twists of the Fermat quartic (31 can be realized over* $\mathbb{Q}$*). These give rise to* 54 *(resp.* 48*) distinct joint (resp. independent) coefficient measures.*
2. *There are* 23 *distinct Sato-Tate groups of twists of the Klein quartic (10 can be realized over* $\mathbb{Q}$*). These give rise to* 23 *(resp.* 22*) distinct joint (resp. independent) coefficient measures.*
3. *There are* 60 *distinct Sato-Tate groups of twists of the Fermat or the Klein quartics (34 can be realized over* $\mathbb{Q}$*). These give rise to* 60 *(resp.* 54*) distinct joint (resp. independent) coefficient measures.*

https://math.mit.edu/~drew/st3/g3SatoTateDistributions.html

The Fermat and the Klein quartics

The **Fermat quartic** is given by the equations:

$$C_1^0\colon x^4 + y^4 + z^4 = 0\,.$$

For the **Klein quartic** $x^3y + y^3z + z^3x = 0$, we will use the special twist:

$$C_7^0\colon x^4 + y^4 + z^4 + 6(xy^3 + yz^3 + zx^3) - 3(x^2y^2 + y^2z^2 + z^2x^2) \\ + 3xyz(x + y + z) = 0\,.$$

We define the following elliptic curves:

$$E_1^0\colon y^2z = x^3 + xz^2\,, \qquad E_7^0\colon y^2z = x^3 - 1715xz^2 + 33614z^3\,,$$

For $d = 1$ or 7, the elliptic curve E_d^0 has CM by $\mathbb{Q}(\sqrt{-d})$.

Proposition

For $d = 1$ or 7, the Jacobian of C_d^0 is $\mathbb{Q}$-isogenous to the cube of E_d^0.

Twists

Let us consider C^0/k and $\mathrm{Aut}(C^0)/M$. We define the **twisting group** $G_{C^0} = \mathrm{Aut}(C^0) \rtimes \mathrm{Gal}(M/k)$. Then,

$$\begin{array}{ccccc} \mathrm{Twist}(C^0/k) & \leftrightarrow & \mathrm{H}^1(G_k, \mathrm{Aut}(C^0)) & \leftrightarrow & \mathrm{Hom}(G_k, G_{G_0}) \\ \phi:\ C \to C^0 & \to & \xi_\sigma = \phi \circ^\sigma \phi^{-1} & \to & \sigma \mapsto (\xi_\sigma, \sigma) \end{array}$$

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In my thesis (2014) I gave an algorithm to compute twists of non-hyperelliptic curves, and I used to compute all twists of plane quartics (non-hyperelliptic genus 3 curves) over number fields.

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We give a explicit **monomorphism**:

$$\iota : \mathrm{Aut}(C^0) \hookrightarrow^{\iota_1} \mathrm{End}(\Omega^1(C^0)) \simeq^{\iota_2} \mathrm{End}(\Omega^1(E^0)^3) \simeq^{\iota_3} \mathrm{End}((E^0)^3) \otimes \mathbb{Q} \hookrightarrow^{\iota_4} \mathrm{GSp}_6(\mathbb{Q})$$

Now, given a twist $C \simeq_L C^0$, it defines a twist

$$\mathrm{Jac}(C^0) \sim_k (E^0)^3 \simeq_L A \sim_k \mathrm{Jac}(C).$$

Moment distributions of $A \sim_L E^3$

Proposition (Fité, L.G., Sutherland)

Let A/k be such that $A_L \sim_L E_L^3$, where E/k is an elliptic curve with CM by M. Suppose that $a_3(\theta)(\tau)$ is rational for every $\tau \in G := \mathrm{Gal}(L/kM)$. For $i = 1, 2, 3$, the sequence $a_i(A_{kM})$ is equidistributed on $I_i = \left[-\binom{2g}{i}, \binom{2g}{i}\right]$ w.r.t. a measure that is continuous up to a finite number of points and then uniquely determined by its moments. For $n \geq 1$, we have $\mathrm{M}_{2n-1}[a_1(A_{kM})] = \mathrm{M}_{2n-1}[a_3(A_{kM})] = 0$ and:

$$
\begin{aligned}
\mathrm{M}_{2n}[a_1(A_{kM})] &= \tfrac{1}{|G|} \textstyle\sum_{\tau \in G} |a_1(\theta)(\tau)|^{2n} \binom{2n}{n}, \\
\mathrm{M}_{n}[a_2(A_{kM})] &= \tfrac{1}{|G|} \textstyle\sum_{\tau \in G} \sum_{i=0}^{n} \binom{n}{i} \binom{2i}{i} |a_2(\theta)(\tau)|^{i} \left(|a_1(\theta)(\tau)|^2 - 2 \cdot |a_2(\theta)(\tau)|\right)^{n-} \\
\mathrm{M}_{2n}[a_3(A_{kM})] &= \tfrac{1}{|G|} \Big(\textstyle\sum_{\tau \in G} \sum_{i=0}^{n} \binom{2n}{2i} \sum_{j=0}^{2i} \sum_{k=0}^{n-i} \binom{2i}{j} (r_1(\tau) - 3)^{2i-j} \\
&\qquad \cdot r_2(\tau)^{2n-2i} \binom{n-i}{k} 4^k (-1)^{n-i-k} \binom{2j+2n-2k}{j+n-k} \Big) .
\end{aligned}
$$

Here $r_1(\tau)$ and $r_2(\tau)$ are the real and imaginary parts of $a_3(\theta)(\tau) a_2(\theta)(\tau) \overline{a}_1(\theta)(\tau)$, respectively.

Moment distributions of $A \sim_L E^3$

In the previous result $\theta := \theta_{M,\sigma}(E, A)$ (resp. $\theta_{M,\sigma}(A)$) stands for the representation afforded by the module $\mathrm{Hom}(E_L, A_L) \otimes_{M,\sigma} \bar{\mathbb{Q}}$ (resp. $\mathrm{End}(A_L) \otimes_{M,\sigma} \bar{\mathbb{Q}}$), and similarly $\overline{\theta} := \theta_{M,\overline{\sigma}}(E, A)$ and $\theta_{M,\overline{\sigma}}(A)$.

★ Idea of the proof:

$$V_\ell(A_{kM}) \simeq V_\sigma(A) \oplus V_{\overline{\sigma}}(A)\,, \qquad V_\ell(E_{kM}) \simeq V_\sigma(E) \oplus V_{\overline{\sigma}}(E)\,.$$

It follows from Theorem 3.1 in Fité's thesis, that

$$V_\sigma(A) \simeq \theta_{M,\sigma}(E, A) \otimes V_\sigma(E)\,, \qquad V_{\overline{\sigma}}(A) \simeq \theta_{M,\overline{\sigma}}(E, A) \otimes V_{\overline{\sigma}}(E)\,.$$

Algebraic Sato-Tate Conjecture

In cases where the **Mumford-Tate conjecture** is explained by endomorphisms and the **twisted decomposable Lefschetz group** over F is connected, the **algebraic Sato-Tate conjecture** is in a sense also explained by endomorphisms. This is the case for $g \leq 3$ and the Sato-Tate group can be computed via the Lefschetz twisted group.

Definition (Lefschetz twisted group)

Given an abelian variety A/k with endomorphism algebra $D = \mathrm{End}^0(A)$ defined over a extension L/k, we define the Lefschetz group of A/k as $\mathrm{L}_k(A) = \bigcup_{\tau \in \mathrm{Gal}(L/k)} \mathrm{L}_k^\tau(A)$, where for each $\tau \in \mathrm{Gal}(L/k)$ we get the component

$$\mathrm{L}_k^\tau(A) = \{\gamma \in \mathrm{Sp}_{2g} : \gamma\beta\gamma^{-1} =^\tau \beta \text{ for all } \beta \in D\}.$$

Theorem (Banaszak, Kedlaya)

For A/k an abelian variety of dimension $g \leq 3$ or with CM, the Sato-Tate group $\mathrm{ST}_k(A)$ is a maximal compact subgroup of $\mathrm{L}_k(A)$.

Genus 3: some distributions

Theorem (Fité, L.G., Sutherland)

The following hold:

1. *There are* 54 *distinct Sato-Tate groups of twists of the Fermat quartic (31 can be realized over* $\mathbb{Q}$*). These give rise to* 54 *(resp.* 48*) distinct joint (resp. independent) coefficient measures.*
2. *There are* 23 *distinct Sato-Tate groups of twists of the Klein quartic (10 can be realized over* $\mathbb{Q}$*). These give rise to* 23 *(resp.* 22*) distinct joint (resp. independent) coefficient measures.*
3. *There are* 60 *distinct Sato-Tate groups of twists of the Fermat or the Klein quartics (34 can be realized over* $\mathbb{Q}$*). These give rise to* 60 *(resp.* 54*) distinct joint (resp. independent) coefficient measures.*

★ Example curves

Full classification for g=3

★ *Sato-Tate groups of abelian threefolds: a preview of the classification.* Francesc Fité, Kiran S. Kedlaya, and Andrew V. Sutherland

★ *Sato-Tate groups of abelian threefolds*. Francesc Fité, Kiran S. Kedlaya, and Andrew V. Sutherland. In ArXiv today!!

There are 410 Sato-Tate groups of abelian threefolds!!

Thanks!