Fourier analysis over commutative algebraic groups and Frobenius distribution

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(Joint work with A. Forey & J. Fresán)
[in progress]

Outline

(1) A general equidistribution theorem

(2) Remarks

(3) A concrete example (lines on a smooth cubic threefold)

Notation : k finite field

$k \subset k_n \subset \bar{k}$, $[k_n : k] = n$

ℓ prime invertible in k, $\iota : \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$

Theorem (Forey – Fresán – K.) Ex. $\left[\sum_{x \in X(k_n)} \chi(x)\, \psi(\mathrm{Tr}_{k_n/k}(f(x)))\right]$

G/k . <u>connected</u> commutative algebraic group

[ex. $\mathbb{G}_a^n$, $\mathbb{G}_m^s$, tori, abelian variety, products ...]

$X_{/k} \subset G$: closed subvariety, irreducible, dim. $= d$

$\mathcal{F}$: lisse ℓ-adic sheaf on X, <u>pure wt. 0</u>

[ex. $\mathcal{L}_{\psi(f)}$, $f : X \longrightarrow \mathbb{A}^1$] $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$ character

$n \geq 1$ $\hat{G}(k_n)$: characters $G(k_n) \longrightarrow \overline{\mathbb{Q}}_\ell^\times \simeq \mathbb{C}^\times$

$t_{\mathcal{F}} : G(k_n) \to \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$

$$S(\mathcal{F}, \chi) = \frac{(-1)^d}{|k_n|^{d/2}} \underbrace{\sum_{x \in X(k_n)}} \chi(x)\, t_{\mathcal{F}}(x; k_n)$$

$\chi \in \hat{G}(k_n)$

"Fourier transform"

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[Ex. $\dfrac{(-1)^d}{|k_n|^{d/2}} \sum_{x \in X(k_n)} \chi(x)\, \psi(\mathrm{Tr}_{k_n/k}(f(x)))$]

Either $S(F, \chi) = 0$ for "most" χ

or there exist $r \geq 0$, $K \subset U_r(\mathbb{C})$ compact group

s.t. for all $f : \mathbb{C} \longrightarrow \mathbb{C}$ continuous / bounded

$$\lim_{N \to \infty} \underbrace{\frac{1}{N} \sum_{n \leq N}}_{\text{(sometimes can be omitted)}} \frac{1}{|\hat{G}(k_n)|} \sum_{\chi \in \hat{G}(k_n)} f(S(F, \chi)) = \int_K f(\mathrm{Tr}\, g)\, dg$$

Prob. Haar measure

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Remarks:

(1) Version at the level of conjugacy classes in $\underline{\underline{K}}$ exists under suitable assumptions (on G, or or $\mathcal{X}$).

(2) Previously known cases:

(a) Deligne's equid. th. $\longrightarrow$ $\widehat{\mathbb{G}_a}$

(b) Katz: $\underline{\underline{\mathbb{G}_m}}$ (2012)

(3) What is the link with Frobenius?

The Lang torsor construction associates to any $\chi \in \hat{G}(k_n)$ a sheaf $\mathcal{L}_\chi$ with trace function $\chi(N_{k_n/k}(x))$, so that

$$S(\mathcal{F},\chi) = \frac{(-1)^d}{|k_n|^{d/2}} \sum_{x \in X(k_n)} \mathrm{Tr}(\mathrm{Fr}_x \mid \mathcal{F} \otimes \mathcal{L}_\chi)$$

(Trace formula)

$$= \frac{(-1)^d}{|k_n|^{d/2}} \sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}(\mathrm{Fr}_{k_n} \mid H^i_c(G_{\bar{k}}, \mathcal{F}\otimes\mathcal{L}_\chi))$$

Deligne's R.H.: eigenvalues α of Fr_{k_n} have $|\alpha| \leq |k_n|^{i/2}$

so the normalization really only works if

$$\begin{cases} H^i_c(\mathcal{F} \otimes \mathcal{L}_\chi) = 0 \quad \text{if } i \neq d \\ H^d_c(\mathcal{F} \otimes \mathcal{L}_\chi) \text{ is "pure" of wt. } d \\ \qquad (|\alpha| = |k_n|^{d/2}) \end{cases}$$

When this occurs:

$$S(\mathcal{F}, \chi) = \frac{1}{|k_n|^{d/2}} \operatorname{Tr}\left(Fr_{k_n} \mid H^d_c(\mathcal{F} \otimes \mathcal{L}_\chi)\right)$$

$=$ <u>trace of a unitary matrix</u>

Tools of the proof

(1) Deligne's RH / formalism

(2) Tannakian formalism / convolution [Katz]

(3) Vanishing thms. for cohomology

$\longrightarrow$ for most x, the previous result works

[Gabber-Loeser, Krämer-Weissauer, ...]

(4) Sawin's Quantitative Sheaf Theory

A concrete example (inspired by Krämer over $\mathbb{C}$)

(char $\neq 2$)

$X/_k \subset \mathbb{P}^4_k$ smooth cubic 3-fold

$F \subset Gr(2,4)$: lines in X [Fano; smooth irred. surface]

$A = Alb(F) \simeq Pic(F)$ ($\simeq_{\mathbb{C}}$ interm. Jacobian)

abelian of dim. 5

$i : F \hookrightarrow A$ [Beauville]

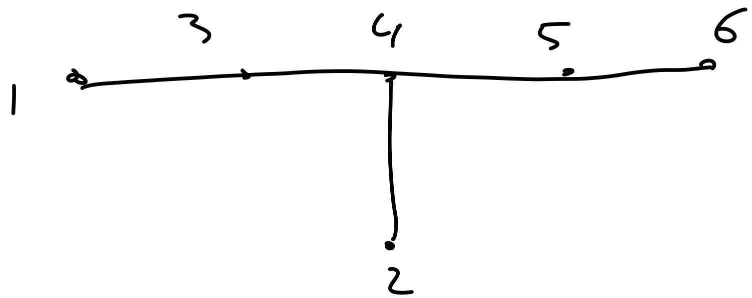
$\mathcal{F}$: trivial sheaf on $i(F)$

$$S(\mathcal{F}, \chi) = \frac{1}{|k_n|} \sum_{x \in F(k_n)} \chi(x)$$

"**Th.**" – The group K in that case has connected component of exceptional type E_6.

Remark. $\dim E_6 = 78$

has a faithful repr. of dim 27



Sketch:

Criterion – Let $K \subset U_{27}(\mathbb{C})$ be connected, semisimple compact group such that

(1) K is ~~not self-dual~~

(2) $M_4(K) = \int_K |\operatorname{Tr} g|^4 \, dg = 3$

Then $K \simeq E_6$ [compact].

The point is to check these; key is (2) which we do by computing

$$M_4(k) = \lim_{N \to \infty} \frac{1}{N} \sum_{n \le N} \frac{1}{|A(\mathbb{F}_n)|} \sum_x |S(\mathcal{F}, x)|^4$$

$$\overset{?}{=} 3$$