From the Birch and Swinnerton-Dyer conjecture to Nagao's conjecture

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Around Frobenius Distributions and Related Topics II
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This work is
with **M Ram. Murty** (co-authored) and
with **Andrew V. Sutherland**(contributed the appendix).

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Introduction

Riemann zeta function

The (Riemann) zeta function is a function of a complex variable which is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges when $\mathrm{Re}(s) > 1$. The functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

shows that there are trivial zeros at $s = -2, -4, -6 \ldots$.

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Von Mangoldt function

The Von Mangoldt function is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k\,; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

It appears in the log derivative of the zeta function :

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{4}$$

We consider the sum

$$\psi(t) = \sum_{n \leq t} \Lambda(n). \tag{5}$$

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[Cramér, 1921]

When Riemann Hypothesis is true, we have

$$\lim_{x \to \infty} \frac{1}{\log x} \int_2^x \left(\frac{\psi(t) - t}{t} \right)^2 dt = \sum_\rho \left| \frac{n_\rho}{\rho} \right|^2, \tag{7}$$

where ρ is nontrivial distinct zeros of $\zeta(s)$, and n_ρ is the multiplicity of the corresponding zero ρ.

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For L function of elliptic curves

Definition

Let E be an elliptic curve over $\mathbb{Q}$ with discriminant Δ_E and conductor N_E. For each prime $p \nmid \Delta_E$, we write the number of points of $E \pmod p$ as

$$N_p := \#E(\mathbb{F}_p) = p + 1 - a_p, \quad (8)$$

where a_p satisfies Hasse's inequality $|a_p| \leq 2\sqrt{p}$. For $p \mid \Delta_E$, we define $a_p = 0, -1,$ or 1 (Depends on the reduction type).

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The L-function attached to E, denoted as $L_E(s)$ is then defined as an Euler product using this datum :

$$L_E(s) = \prod_{p \mid \Delta_E} \left(1 - \frac{a_p}{p^s}\right)^{-1} \prod_{p \nmid \Delta_E} \left(1 - \frac{a_p}{p^s} + \frac{p}{p^{2s}}\right)^{-1}, \qquad (9)$$

which converges absolutely for $\mathrm{Re}(s) > 3/2$ by virtue of Hasse's inequality.

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By the work of Wiles, and Breuil, Conrad, Diamond, and Taylor, $L_E(s)$ extends to an entire function and satisfies a functional equation relating $L_E(s)$ to $L_E(2-s)$.

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Analogously, we consider the log derivative of $L_E(s)$:

$$-\frac{L_E'(s)}{L_E(s)} = \sum_{n=1}^{\infty} \frac{c_n \Lambda(n)}{n^s},$$

where

$$c_n = \begin{cases} \alpha_p^m + \beta_p^m, & \text{if } n = p^m \text{ and } p \nmid N, \\ a_p^m, & \text{if } n = p^m \text{ and } p \mid N, \\ 0, & \text{otherwise.} \end{cases}$$

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Cramér-type estimation for L function of elliptic curves

With

$$\psi_E(t) = \sum_{n \le t} c_n \Lambda(n), \tag{11}$$

[Kim-Murty, 2020]

Assuming the Riemann hypothesis for $L_E(s)$ is true, we obtain

$$\lim_{x\to\infty} \frac{1}{\log x} \int_2^x \frac{\psi_E^2(t)}{t^3} dt = \sum_\rho \left| \frac{n_\rho}{\rho} \right|^2, \tag{12}$$

where the sum is over all nontrivial distinct zeros ρ of $L_E(s)$, and n_ρ is the multiplicity of each ρ.

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Implications of the estimation ?

The Birch and Swinnerton-Dyer conjecture

A version of BSD

For some constant C_E, we have

$$\prod_{\substack{p<x \\ p\nmid\Delta_E}} \frac{N_p}{p} \sim C_E(\log x)^r, \tag{13}$$

where r is the order of the zero of the L-function $L_E(s)$ of E at $s = 1$.

Furthermore, Birch and Swinnerton-Dyer conjectured that the order of the zero of the L-function $L_E(s)$ is equal to the rank of the Mordell-Weil group $E(\mathbb{Q})$.

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The BSD conjecture is true if and only if

$$\sum_{\substack{p^k \le x \\ p \nmid \Delta_E}} (\alpha_p^k + \beta_p^k) \log p = o(x \log x).$$

Whereas, the Riemann hypothesis for $L_E(s)$ is equivalent to

$$\sum_{\substack{p^k \le x \\ p \nmid \Delta_E}} (\alpha_p^k + \beta_p^k) \log p = \mathcal{O}(x (\log x)^2).$$

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By applying Perron's formula, and considering the subsequential limit, we obtain the following :

[Kim-Murty, 2020]

Assume the Riemann hypothesis is true for $L_E(s)$. Then there is a sequence $x_n \in [2^n, 2^{n+1}]$ such that

$$\lim_{n\to\infty} \frac{1}{\log x_n} \sum_{p<x_n} \frac{a_p \log p}{p} = -r + \frac{1}{2}, \tag{14}$$

where r is the order of $L_E(s)$ at $s = 1$.

[Kim-Murty, 2020]

If the limit

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p<x} \frac{a_p \log p}{p} \tag{15}$$

exists, then the Riemann hypothesis for $L_E(s)$ is true, and the limit is $-r+1/2$ (Nagao-Mestre sum).

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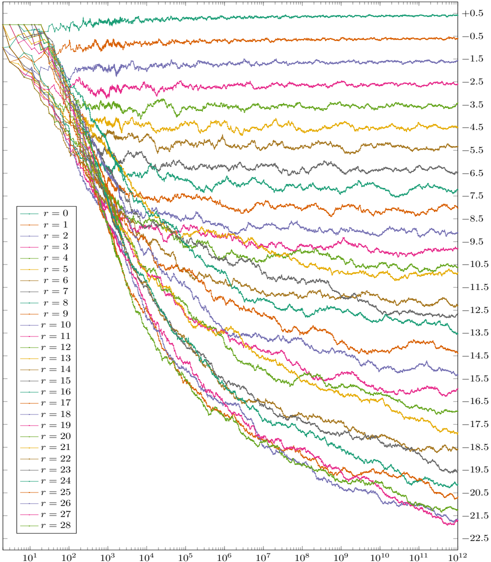
Experimental data by Andrew V. Sutherland

We define

$$S(x) := \frac{1}{\log x} \sum_{p \leq x,\ p \nmid \Delta_E} \frac{a_p(E) \log p}{p}.$$

We expect this sum converges to $-r + \frac{1}{2}$ as $x \to \infty$.

By Andrew V. Sutherland



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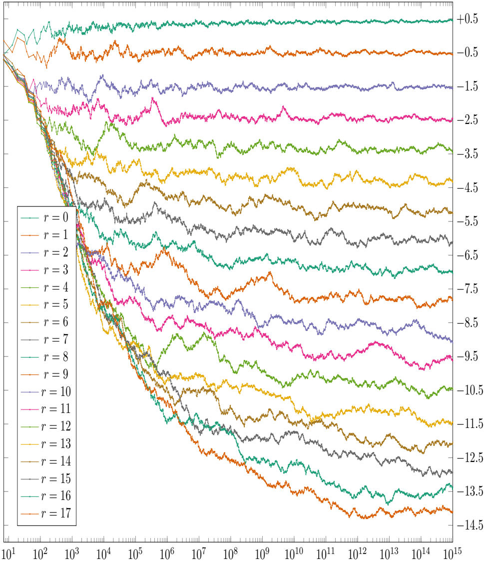

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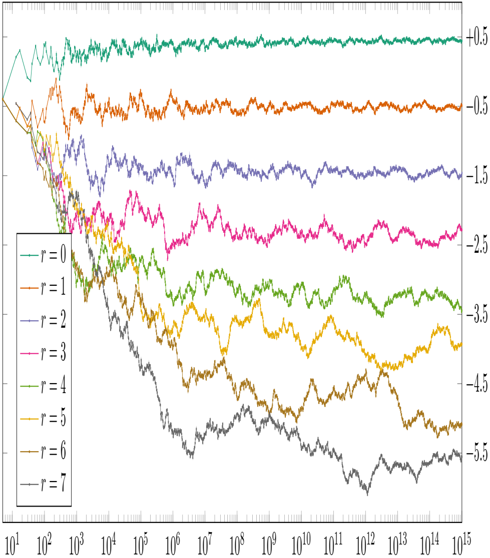

Sketch of the proof :

With

$$\psi_E(t) = \sum_{n \le t} c_n \Lambda(n), \tag{17}$$

We want to prove

Assuming the Riemann hypothesis for $L_E(s)$ is true, we obtain

$$\lim_{x \to \infty} \frac{1}{\log x} \int_2^x \frac{\psi_E^2(t)}{t^3} dt = \sum_{\rho} \left| \frac{n_\rho}{\rho} \right|^2, \tag{18}$$

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Sketch of the proof for (27)

Using Cauchy Residue theorem, and Perron's formula with careful error estimation, we obtain

$$\frac{\psi_E^2(t)}{t^3} = \frac{1}{t^3}\left(\sum_{\rho} n_\rho \frac{t^\rho}{\rho} + \mathcal{O}\left(\frac{t^2}{R}\right)\right)^2, \tag{19}$$

and by choosing sufficiently big R (size of the domain in Cauchy Residue theorem),

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$$= \log x \sum_\rho \left|\frac{n_\rho}{\rho}\right|^2 + \sum_\rho \frac{n_\rho}{\rho} \sum_{\rho'\neq 2-\rho} \frac{n_{\rho'}}{\rho'} \cdot \frac{x^{\rho+\rho'-2} - 2^{\rho+\rho'-2}}{\rho+\rho'-2} + \mathcal{O}(1),$$

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can be estimated by considering the two separate cases :

$$|\rho+\rho'-2| \geq \eta \quad \text{and} \quad |\rho+\rho'-2| < \eta.$$

and by following the result on the number of zeros of $L_E(s)$ in a bounded region :

Theorem (Selberg)

The number of zeros such that $0 < \mathrm{Im}(\rho) \leq T$ of $L_E(s)$ satisfies

$$N_E(T) = \frac{\alpha_E}{\pi} T(\log T + c) + \mathcal{O}(\log T), \tag{20}$$

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Nagao's conjecture

For an elliptic curve $\mathcal{E}$ over $\mathbb{Q}(T)$, at each prime p and $T = t$

$$a_p(\tilde{\mathcal{E}}_t) = 1 - \#\tilde{\mathcal{E}}_t(\mathbb{F}_p) + p,$$

and define a fibral average of the trace of Frobenius for each p :

$$A_p(\mathcal{E}) = \frac{1}{p}\sum_{t=0}^{p-1} a_p(\tilde{\mathcal{E}}_t).$$

Nagao's Conjecture for elliptic surfaces over $\mathbb{Q}$ (1997)

$$\lim_{X\to\infty} \frac{1}{X}\sum_{p\leq X} -A_p(\mathcal{E}) \log p = \operatorname{rank} \mathcal{E}(\mathbb{Q}(T)).$$

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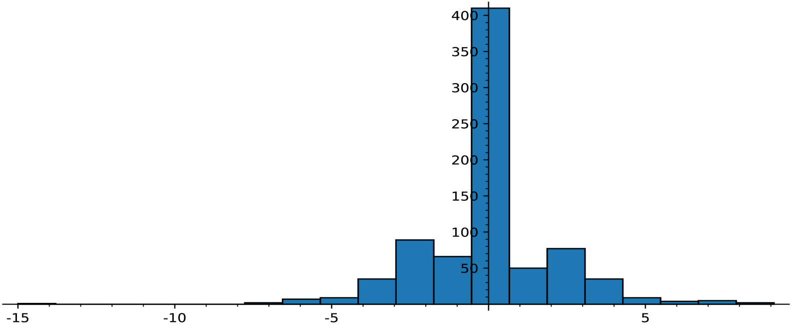
FIGURE – $A_p(\mathcal{E})$ of $\mathcal{E} : y^2 = x^3 - 3t^4x - t^2(3 + 2t^8)$
x axis : $A_p(\mathcal{E})$
y axis : Frequency of $A_p(\mathcal{E})$ up to 500th prime p



It appears that the average of $A_p(\mathcal{E})$ is 0, indeed $\mathcal{E}(\mathbb{Q}(T))$ has rank 0.

A generalization of Nagao's conjecture

Let $\mathcal{X} \to C$ be a surface with a fibration to a curve with generic fiber $X/k(C)$. For each prime $\mathfrak{p}$, we have $\tilde{\mathcal{X}} \to \tilde{C}$ and define a fibral average of the trace of Frobenius :

$$A_{\mathfrak{p}}(\mathcal{X}) = \frac{1}{q_{\mathfrak{p}}} \sum_{c \in \tilde{C}(\mathbb{F}_{\mathfrak{p}})} a_{\mathfrak{p}}(\tilde{\mathcal{X}}_c), \quad q_{\mathfrak{p}} = \#\mathbb{F}_{\mathfrak{p}}.$$

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Rewriting the Nagao conjecture

Changing the order of the sum, Nagao's conjecture becomes

$$-\frac{1}{N}\sum_{p\leq N} A_p(\mathcal{E})\log p = -\sum_{p\leq X}\frac{1}{p}\sum_{t\leq p} a_p(\mathcal{E}_t)\log p$$

$$= -\sum_{t\leq X}\left(\sum_{t\leq p\leq X}\frac{a_p(\mathcal{E}_t)\log p}{p}\right),$$

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If the limit

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Therefore, from our analysis, for a fixed t, the inner sum is **(ignoring error terms)**

$$\left(-r_t + \frac{1}{2}\right) \log \frac{X}{t},$$

for $r_t = \text{rank } \mathcal{E}_t(\mathbb{Q})$. So one may expect

$$-\frac{1}{X} \sum_{p \le X} A_p(\mathcal{E}) \log p \sim \frac{1}{X} \sum_{t \le X} \left(r_t - \frac{1}{2}\right) \log \frac{X}{t}.$$

This suggests the following (modified) form of Nagao's conjecture :

$$\lim_{X \to \infty} \frac{1}{X} \sum_{t \le X} \left(r_t - \frac{1}{2}\right) \log \frac{X}{t} = \text{rank } \mathcal{E}(\mathbb{Q}(T)). \qquad (22)$$

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Using Stirling's formula, we get

$$\frac{1}{X}\sum_{t\leq X} r_t \sim \left(\text{rank } \mathcal{E}(\mathbb{Q}(T)) + \frac{1}{2}\right) \tag{23}$$

as $X \to \infty$ **(ignoring error terms)**.

Experimental results

Fermigier studied 66918 curves lying in 93 families having generic ranks between 0 to 4, and he found

$$\operatorname{rank}\mathcal{E}_t(\mathbb{Q}) = \operatorname{rank}\mathcal{E}(\mathbb{Q}(T)) + \begin{cases} 0 & \text{with probability } 32\%, \\ 1 & \text{with probability } 48\%, \\ 2 & \text{with probability } 18\%, \\ 3 & \text{with probabiity } 2\%. \end{cases}$$

Hence, for every family of elliptic curves which is considered, Fermigier found the quantity

$$\frac{1}{2X}\sum_{|t|\leq X} \operatorname{rank}\ \mathcal{E}_t(\mathbb{Q}) - \operatorname{rank}\ \mathcal{E}(\mathbb{Q}(T)) - \frac{1}{2}$$

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Based on the heuristic results

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Interesting example

Recall the functional equation of $L_E(s)$: For $E/\mathbb{Q}$, we have

$$\Lambda_E(2-s) = W(E)\Lambda_E(s),$$

where $\Lambda_E(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s)$, and $W(E) \in \{\pm 1\}$ is the root number of E. Then it is conjectured that

$$W(E) = (-1)^{\operatorname{rank} E(\mathbb{Q})} \quad \text{(Parity conjecture).} \tag{24}$$

Silverman suggested the following example of Rizzo :

Consider $\mathcal{E} : y^2 = x^3 + Tx^2 - (T+3)x + 1$, then $j(T) = 256(T^2 + 3T + 9)$ and it is an elliptic curve defined over $\mathbb{Q}(T)$ which has rank $\mathcal{E}(\mathbb{Q}(T)) = 1$, while the root number of each fiber

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via the specialization theorem, one cannot expect that the rank would jump by 2 very often. and cannot jump by 1 by the sign of the functional equation. Hence, we expect

$$\lim_{X \to \infty} \frac{1}{2X} \sum_{|t| \leq X} \operatorname{rank} \mathcal{E}_t(\mathbb{Q}) = \operatorname{rank} \mathcal{E}(\mathbb{Q}(T)).$$

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Hence, there would not be extra 1/2 coming from the "fact" that half of the fibers have the wrong negative sign (root number) in their functional equation. Hence, we need to reformulate

$$\frac{1}{2X}\sum_{|t|<X} \text{rank } \mathcal{E}_t(\mathbb{Q}) \to \text{rank}\,\mathcal{E}(\mathbb{Q}(T)) + S(E)$$

as $X \to \infty$, and $S(E)$ is some sort of average of the signs appearing in functional equations of the fibers.

Conjecture by Kim and Murty

For $r_t = \text{rank } \mathcal{E}_t(\mathbb{Q})$, we conjecture : define

$$\mathcal{T} := \left\{ t \in \mathbb{Z} : W(\mathcal{E}_t) = (-1)^{\text{rank}\, \mathcal{E}(\mathbb{Q}(T))+1} \right\}. \tag{25}$$

Then we have

$$\lim_{X\to\infty} \frac{1}{2X} \sum_{|t|\leq X} r_t = \text{rank } \mathcal{E}(\mathbb{Q}(T)) + \delta(\mathcal{T}) \tag{26}$$

where $\delta(\mathcal{T})$ is the natural density of $\mathcal{T}$ as a subset of $\mathbb{Z}$.

What is known ?

Assume that the limit

$$\lim_{X\to\infty} \frac{1}{2X} \sum_{|t|\leq X} r_t$$

exists. Then the parity conjecture (for each fiber of $\mathcal{E}$) implies that the Conjecture is true modulo 2.

Selberg Class

The Selberg class $\mathbb{S}$ consists of complex variable functions $F(s)$ satisfying the following properties :

1. For $\Re(s) > 1$, we can write $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with $a_1 = 1$.
2. For some integer $m \geq 0$, $(s-1)^m F(s)$ extends to an entire function of finite order.
3. There are numbers $Q > 0$, $\alpha_i > 0$, $r_i \in \mathbb{C}$ with $\Re(r_i) \geq 0$ such that
$$\Phi(s) = Q^s \prod_{i=1}^{\infty} \Gamma(\alpha_i s + r_i) F(s)$$
which satisfies a functional equation $\Phi(s) = w\overline{\Phi}(1-s)$ with a complex number $|w| = 1$ and $\overline{\Phi}(s) := \overline{\Phi(\overline{s})}$.
4. There exists an Euler product $F(s) = \prod_p F_p(s)$ for $\Re(s) > 1$.
5. For any fixed $\epsilon > 0$, $a_n = \mathcal{O}_\epsilon(n^\epsilon)$.

Examples

There are well-known examples of elements in $\mathbb{S}$.

1. The Riemann zeta function $\zeta(s) \in \mathbb{S}$.
2. Dirichlet L functions $L(s, \chi)$ and their vertical shifts $L(s + i\theta, \chi)$ are in $\mathbb{S}$.
3. For a number field K, the Dedekind zeta function $\zeta_K(s)$ is in $\mathbb{S}$.
4. For a holomorphic newform f, the associated L-function $L(s, f)$ is in $\mathbb{S}$.

$$\vdots$$

A generalization for the Selberg class

We can extend the previous theorem to L-functions in the Selberg class. Let $F(s) \in \mathbb{S}$ and write

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s} \text{ and } \psi_F(t) := \sum_{n \le t} \Lambda_F(n).$$

A generalization

Assuming the Riemann hypothesis, if it can be formulated, for $F(s)$ is true, we obtain

$$\lim_{x \to \infty} \frac{1}{\log x} \int_2^x \frac{\psi_F^2(t)}{t^2} dt = \sum_{\rho} \left| \frac{n_\rho}{\rho} \right|^2, \tag{27}$$

where the sum is over all nontrivial distinct zeros ρ of $F(s)$, and n_ρ is the multiplicity of each ρ.

Hence, if we write r as the order of $F(s)$ at $s = \frac{1}{2}$, we expect

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p<x} \frac{\Lambda_F(p)}{p} = -r + A,$$

where A is a constant. We (conjecturally) guess that $A = \frac{1}{2}$ as well for a primitive $F \in \mathbb{S}$ by the conjectural theory of the Rankin-Selberg convolution for the Selberg class $\mathbb{S}$.

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