# Towards the motivic Nagao's conjecture and its connections with the Tate conjectures

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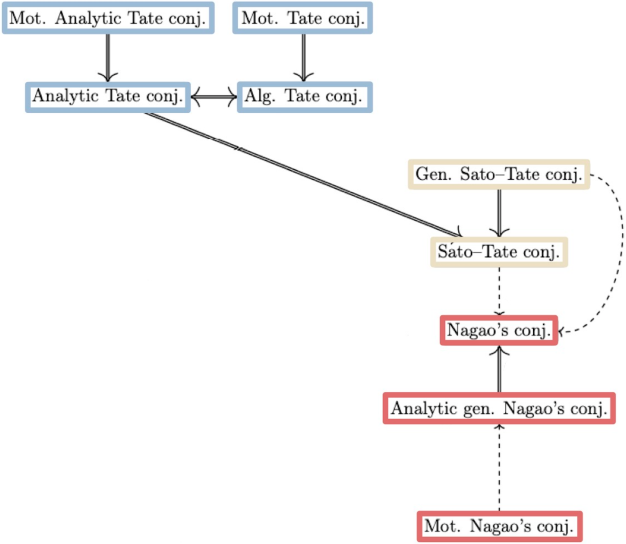
GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Jun. 28-29, 2021

Conference: Around Frobenius Distributions and Related Topics II
Organizers: Alina Carmen Cojocaru and Francesc Fité

# Road map


Mot. Analytic Tate conj.
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Alg. Tate conj.
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Nagao's conj.
Analytic gen. Nagao's conj.
Mot. Nagao's conj.

# Motives

- $k$ – a field of $char(k) = 0$;
- $\mathrm{Mot}_k$ – the Tannakian, semi-simple, graded and polarizable category of pure motives in the sense of Y. André;
- $M = (X, p, n) \in \mathrm{Ob}(\mathrm{Mot}_k)$ – a pure motif where
    - $X$ – is a smooth projective variety defined over $k$,
    - $p$ – is an algebraic correspondence of $X \times X$ which is an idempotent element,
    - $n$ – an integer which is called a *Tate twist*.

# Motives

- $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$;
- $\sigma : k \hookrightarrow \mathbb{C}$;
- $\ell$ a prime number.

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Cat. of Hodge structures

Cat. of Motives in the sense of Y. André

Cat. of continuous representations of $\Gamma_k$

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$$\begin{tikzcd}
\mathrm{HS}_{\mathbb{Q}} \arrow[d, "u_\sigma"'] & \mathrm{Mot}_k \arrow[l, "r_\sigma"'] \arrow[r, "r_\ell"] & \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k) \arrow[d, "u_\ell"] \\
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# Tannakian groups

## Definition

$\mathcal{G}_k := \mathrm{Aut}^{\otimes}(\omega)$ - Motivic Galois group,
$G_{\ell,k} := \mathrm{Aut}^{\otimes}(u_{\ell|im(r_\ell)})$ - $\ell$-adic monodromy group,
$G_\sigma := \mathrm{Aut}^{\otimes}(u_\sigma)$ - Big Mumford–Tate group.

## Definition

For every motive $M \in \mathrm{Ob}(\mathrm{Mot}_k)$ we define:
$\mathcal{G}_k(M) := \mathrm{Aut}^{\otimes}(\omega_{|<M>})$,
$G_{\ell,k}(M) := \mathrm{Aut}^{\otimes}(u_{\ell|<r_\ell(M)>})$,
$G_\sigma(M) := \mathrm{Aut}^{\otimes}(u_{\sigma|<r_\sigma(M)>})$.

## Remark

$$G_{\ell,k}(M) = \overline{im(\Gamma_k \to \mathrm{GL}(r_\ell(M)))}^{\mathrm{Zar}}.$$

# Motivic $L$-functions

- $k$ – a number field;
- $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$, the inertia $I_\mathfrak{p}$ and decomposition $D_\mathfrak{p}$ subgroups of $\Gamma_k$;
- $\mathrm{Frob}_\mathfrak{p}$ – the Frobenius element.

## Definition

We define the Euler factor of $M$ at $\mathfrak{p}$ for every integer $j$ as follows

$$L_{\mathfrak{p},j}(s, M) := det(1 - q_j^{-s}\,\mathrm{Frob}_\mathfrak{p} \mid (\wedge^j r_\ell(M))^{I_\mathfrak{p}} \otimes_{\mathbb{Q}_\ell} \mathbb{C})^{-1},$$

where $q_\mathfrak{p}$ is the norm of $\mathfrak{p}$.

# Motivic $L$-function

Let us remark that when $M$ has good reduction at $\mathfrak{p}$

- it is known that $L_{\mathfrak{p},j}(s, M)$ does not depend on the choice of $D_{\mathfrak{p}}$,
- it is conjectured that $L_{\mathfrak{p},j}(s, M)$ does not depend on the choice of the embedding $\iota : \mathbb{Q}_\ell \to \mathbb{C}$ nor on the prime number $\ell$.

## Definition

Ordinary $L$-function of $M$

$$L_j(s, M) := \prod_{\mathfrak{p}} L_{\mathfrak{p},j}(s, M).$$

# Algebraic Tate conjectures for abelian varieties

- $k$ – a number field;
- $A$ – an abelian variety defined over $k$ of dimension $g$;
- $\ell$ a prime number.

### Algebraic Tate conjecture '63

Every **Tate class** is a $\mathbb{Q}_\ell$-linear combination of algebraic classes.

# Algebraic Tate conjectures for abelian varieties

- $T_\ell(A)$ – the $\ell$-adic Tate module,
- $\rho_\ell : \Gamma_k \to \mathrm{Aut}(T_\ell(A)) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell) \subset \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$

**Definition**

The $\ell$-adic monodromy group is an algebraic over $\mathbb{Q}_\ell$ defined as $\mathrm{G}_\ell(A) := \overline{\rho_\ell(\Gamma_k)}^{Zar}$.

**Definition**

The group of **Tate classes** of codimension $p$ is define as follows

$$\mathcal{T}^p := H^{2p}_{\acute{e}t}(A, \mathbb{Q}_\ell)^{\mathrm{G}_\ell(A)}.$$

# Tate conjectures for smooth projective varieties

- $k$ – a finitely generated field with absolute Galois group $\Gamma_k$,
- $X$ – a smooth projective variety defined over k,
- $\ell$ – a prime number $\ell \neq \mathrm{char}(k)$.

Consider the cycle map:

$$cl^j : Z^j(X) \to H^{2j}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell(j))^{\Gamma_k}.$$

# Tate conjectures for smooth projective varieties

## Algebraic Tate conjecture

$A^j(X) := \mathrm{Im}\ cl^j$ is finitely generated, and we have the following isomorphism

$$A^j(X) \otimes \mathbb{Q}_\ell \to H^{2j}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell(j))^{\Gamma_k};$$

## Analytic Tate conjecture

The rank of the image of the cycle map $A^j(X)$ is equal to the order of pole of the $L$-function of $H^{2j}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell)$ at $s = j + \dim(k)$, where $\dim(k)$ is the transcendence degree of the field $k$.

# Motivic algebraic and analytic Tate conjectures

## Motivic algebraic Tate Conjecture

Let $M \in \mathrm{Ob}(\mathrm{Mot}_k)$, then for every prime number $\ell$ we have

$$G_{\ell,k}(M) = \mathcal{G}_k(M)_{\mathbb{Q}_\ell}.$$

## Motivic analytic Tate Conjecture

Let $M = (X, p, n) \in \mathrm{Ob}(\mathrm{Mot}_k)$ be a motive of weight 1, where $X$ is a smooth projective variety over the finitely generated field $k$. Then, for every integer $j$ the rank of $A^j(X)$ is equal to the order of pole of the ordinary $L$-function $L_{2j}(s, M)$ at $s = j + \dim_a(k)$.

## What is the relation between the algebraic and analytic Tate conjectures?

**Theorem (C.-F., Kim 2020)**

*Let $k$ be a finitely generated field,* **then** *the algebraic Tate conjecture and analytic Tate conjecture are equivalent.*

# Does the motivic algebraic Tate conjecture implies the algebraic Tate conjecture?

## Motivic algebraic Tate Conjecture

Let $M \in \mathrm{Ob}(\mathrm{Mot}_k)$, then for every prime number $\ell$ we have

$$G_{\ell,k}(M) = \mathcal{G}_k(M)_{\mathbb{Q}_\ell}.$$

## Theorem (C.-F., Kim 2020)

*Let $X$ be an abelian variety defined over a number field $k$ and let $M$ be the abelian motive associated to $X$. Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and assume that $X_\sigma$ satisfies Hodge conjecture,* **then** *the motivic Tate conjecture for $M$ implies the algebraic Tate conjecture for $X$.*

# Idea of the proof

- Let $X/k$ be an abelian variety defined over a number field $k$ of dimension $g$.
- The algebraic Tate conjecture is equivalent to the following equality for every integer $1 \leq j \leq g$:

$$H^{2j}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell(j))^{G_{\ell,k}(X)^\circ} = A^j(X)_{\mathbb{Q}_\ell}. \quad (1)$$

- We know that for every abelian motive $M$, the motivic Tate conjecture is equivalent to the motivic Mumford–Tate conjecture (C.-F., Commelin 2019).
- Hence, let $M$ be the abelian motive associated to $X$, we know that

$$G_{\ell,k}(M) = \mathcal{G}_k(M)_{\mathbb{Q}_\ell} \iff G^\circ_{\ell,k}(M) = G_\sigma(M)_{\mathbb{Q}_\ell}. \quad (2)$$

- In particular, we have the following equalities:

$$G_{\ell,k}(M)^\circ = \mathcal{G}_k(M)^\circ_{\mathbb{Q}_\ell} = G_\sigma(M)_{\mathbb{Q}_\ell}. \quad (3)$$

# Idea of the proof

- Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and assume that $X_\sigma$ satisfies Hodge conjecture. Then for every integer $1 \le j \le g$ we have:

$$H_B^{2j}(X_\sigma, \mathbb{Q})^{G_\sigma(X_\sigma)} = A^j(X_\sigma). \quad (4)$$

- From equation (3) we have for every integer $1 \le j \le g$:

$$H_{\acute{e}t}^{2j}(X_{\bar{k}}, \mathbb{Q}_\ell(j))^{G_{\ell,k}(X)^\circ} = A^j(X)_{\mathbb{Q}_\ell}. \quad (5)$$

- From (1) we proved that the motivic Tate conjecture for the associated abelian motive $M$, implies the algebraic Tate conjecture for the abelian variety $X$.

# Does the motivic analytic Tate conjecture implies the analytic Tate conjecture?

## Motivic analytic Tate Conjecture

Let $M = (X, p, n) \in \mathrm{Ob}(\mathrm{Mot}_k)$ be a motive of weight 1, where $X$ is a smooth projective variety over the finitely generated field $k$. Then, for every integer $j$ the rank of $A^j(X)$ is equal to the order of pole of the ordinary $L$-function $L_{2j}(s, M)$ at $s = j + \dim_a(k)$.

## Theorem (C.-F., Kim 2020)

*The motivic analytic Tate conjecture implies the analytic Tate conjecture.*

# From the algebraic Tate conjecture to the Sato–Tate conjecture

**Theorem (Tate '63)**

*Let $E$ be an elliptic curve defined over a number field $k$ without complex multiplication. Then the analytic Tate conjecture for $E^m$ implies the Sato–Tate conjecture for $E$.*

## From the algebraic Tate conjecture to the Sato–Tate conjecture

**Theorem (Tate '63 + $\varepsilon$)**

*Let $E$ be an elliptic curve defined over a number field $k$ without complex multiplication. Then the analytic Tate conjecture for $E^m$ implies the Sato–Tate conjecture for $E$, and* **hence**, *the algebraic Tate conjecture for $E^m$ implies the Sato–Tate conjecture for $E$.*

# From the Sato–Tate conjecture to Nagao's conjecture

**Theorem (Kim 2018)**

*The generalized Sato–Tate conjecture partially implies the Nagao's conjecture for families of hyperelliptic curves defined by quadratic twists.*

# Towards the motivic Nagao's conjecture

## Nagao's heuristic

Relate the rank of a nonsplit elliptic curves defined over $\mathbb{Q}(T)$ with a limit formula arising from a weighted average of Frobenius traces from each fiber.

**Setting:**

- $\mathcal{E}$: an elliptic curve defined over $\mathbb{Q}(T)$ with Weierstraß equation

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T) \qquad A(T),\ B(T) \in \mathbb{Z}[T].$$

- $\Delta_{\mathcal{E}} = 4A(T)^3 + 27B(T)^2 \neq 0.$

# Towards the motivic Nagao's conjecture

## Definition (Frobenius trace of each fiber)

For every prime number $p$ and any point $t \in \mathbb{Z}$ we have

$$a_p(\mathcal{E}_t) = p + 1 - |\mathcal{E}_t(\mathbb{F}_p)|.$$

## Definition (Fibral average of Frobenius traces at $p$)

$$A_p(\mathcal{E}) = \frac{1}{p} \sum_{t=0}^{p-1} a_p(\mathcal{E}_t).$$

## Nagao's conjecture 1997

$$-\lim_{X \to \infty} \frac{1}{X} \sum_{p \leq X} A_p(\mathcal{E}) \log p = \mathrm{rank}\mathcal{E}(\mathbb{Q}(T)).$$

## Towards the motivic Nagao's conjecture

- $k$ – be a number field;
- $M \in \mathrm{Ob}(\mathrm{Mot}_k)$ – a pure motive with Euler factor at $\mathfrak{p}$ given by

$$L_{\mathfrak{p},j}(s,M) := det(1 - q_j^{-s}\,\mathrm{Frob}_{\mathfrak{p}} \mid (\wedge^j r_\ell(M))^{I_{\mathfrak{p}}} \otimes_{\mathbb{Q}_\ell} \mathbb{C})^{-1};$$

- The local coefficients of $L_{\mathfrak{p},j}(s,M)$ are given by

$$a_{\mathfrak{p},2j}(M) := Tr(\mathrm{Frob}_{\mathfrak{p}} \mid (\wedge^{2j} r_\ell(M))^{I_{\mathfrak{p}}} \otimes_{\mathbb{Q}_\ell} \mathbb{C}), \quad j = 0, \ldots, \dim(X).$$

# Towards the motivic Nagao's conjecture

| Nagao | Motivic Nagao |
|---|---|
| $\mathcal{E}/\mathbb{Q}(T)$ | $\mathcal{M} \in \mathrm{Ob}(\mathrm{Mot}_{k(C)})$ |
| $\forall p, \forall t \in \mathbb{Z}$<br>$\mathcal{E}_t/\mathbb{F}_p$ | $\forall \mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k), \forall t \in C$<br>$\mathcal{M}_t \in \mathrm{Ob}(\mathrm{Mot}_k)$ |
| $a_p(\mathcal{E}_t)$ | $a_{\mathfrak{p},2j}(\mathcal{M}_t)$ |

# Towards the motivic Nagao's conjecture

## Definition (Fibral average of Frobenius traces at $\mathfrak{p}$)

$$\mathcal{A}_{\mathfrak{p},2j}(\mathcal{M}) = \frac{1}{q_{\mathfrak{p}}} \sum_{t \in \tilde{C}(\mathbb{F}_p)} a_{\mathfrak{p},2j}(\mathcal{M}_t).$$

## Motivic Nagao conjecture [C.-F., Kim 2020]

For all integer $j$ we have

$$\lim_{X \to \infty} \frac{1}{X} \sum_{q_{\mathfrak{p}} \leq X} \mathcal{A}_{\mathfrak{p},2j}(\mathcal{M}) \log q_{\mathfrak{p}} = \operatorname{rank} A^j(\mathcal{M}).$$

# Towards the motivic Nagao's conjecture

| Nagao | Motivic Nagao |
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| $\mathcal{E}/\mathbb{Q}(T)$ | $\mathcal{M} \in \mathrm{Ob}(\mathrm{Mot}_{k(C)})$ |
| $\forall p, \forall t \in \mathbb{Z}$<br>$\mathcal{E}_t/\mathbb{F}_p$ | $\forall \mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k), \forall t \in C$<br>$\mathcal{M}_t \in \mathrm{Ob}(\mathrm{Mot}_k)$ |
| $a_p(\mathcal{E}_t)$ | $a_{\mathfrak{p},2j}(\mathcal{M}_t)$ |
| Fix $p$<br>$A_p(\mathcal{E})$ | Fix $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$<br>$\mathcal{A}_{\mathfrak{p},2j}(\mathcal{M})$ |

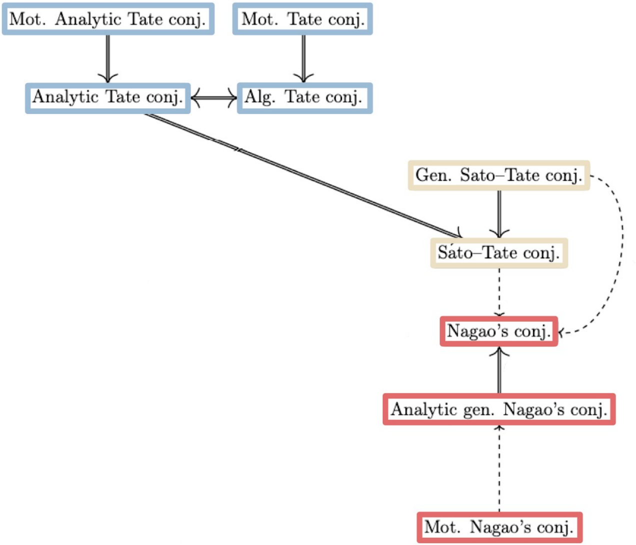
Nagao's conjecture

$$-\lim_{X\to\infty} \frac{1}{X} \sum_{p\leq X} A_p(\mathcal{E}) \log p = \mathrm{rank}\mathcal{E}(\mathbb{Q}(T))$$

Motivic Nagao conjecture

$$\lim_{X\to\infty} \frac{1}{X} \sum_{q_{\mathfrak{p}}\leq X} \mathcal{A}_{\mathfrak{p},2j}(\mathcal{M}) \log q_{\mathfrak{p}} = \mathrm{rank}A^j(\mathcal{M})$$

# Road map



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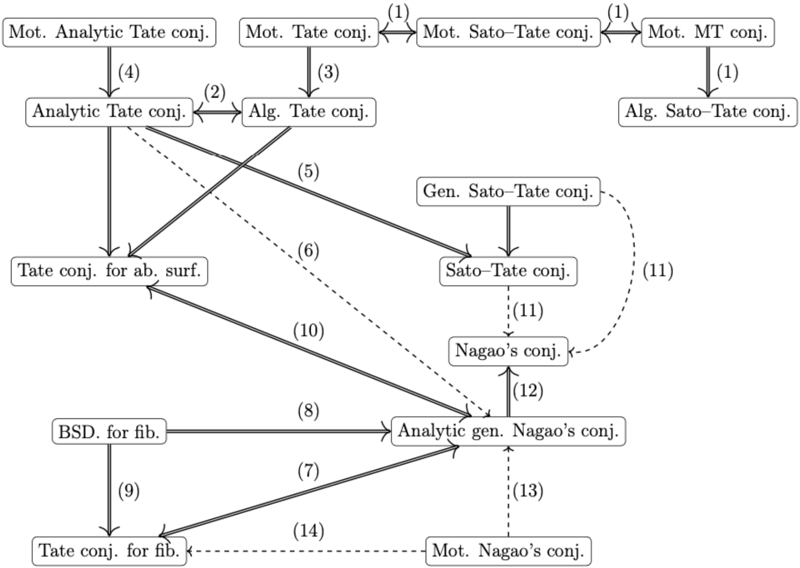
# Relations among conjectures

FIGURE 1. Bridges between Tate conjectures



# Questions?

*"Papers should include more side remarks, open questions, and such. Very often, these are more interesting than the theorems actually proved. Alas, most people are afraid to admit that they don't know the answer to some question, and as a consequence they refrain from mentioning the question, even if it is a very natural one. What a pity! As for myself, I enjoy saying 'I do not know'."*

*– J.-P. Serre*

Thank you!