Zeta statistics

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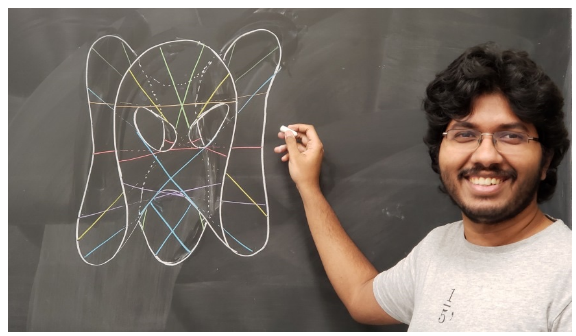

General question:

$(X_n)_{n \geq 1}$ vars $/\mathbb{F}_q$

$$Z_{X_n}(q^{-\dim X_n} t) \xrightarrow{?} \quad ?$$

The modified Grothendieck ring of varieties

k field

$$K_0(\mathrm{Var}_k) = \mathbb{Z}\langle \text{iso classes of var}/k \rangle \Big/ \; X - Z - U$$

$(X \text{ var}/k,\ Z \hookrightarrow X,\ U = X \setminus Z)$

$X - Y$

$(f : X \to Y$ radicial surjective)

bijective + induces insep ext of res fields

Product: $[X] \cdot [Y] = [X \times Y]$ $\qquad 1 = [\operatorname{Spec} k]$

Element $\mathbb{L}$: $\mathbb{L} = [\mathbb{A}^1_k]$

Localised Grothendieck ring:

$$\mathcal{M}_k = K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$$

Dimensional filtration:

$\mathrm{Fil}^d \mathcal{M}_k$ = subgp of $\mathcal{M}_k$ generated by
$[X]\mathbb{L}^{-n}$, $\dim X - n \leq -d$

Decreasing, exhaustive filtration on $\mathcal{M}_k$

Completion : $\widehat{\mathcal{M}_k}$.

Counting measure

$k = \mathbb{F}_q$

$$\#_{\mathbb{F}_q} : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$$
$$X \mapsto |X(\mathbb{F}_q)|$$

Question: $(a_n)_n$ seq of elements of $\mathcal{M}_{\mathbb{F}_q}$

How to compare

1. $\#_{\mathbb{F}_q}(a_n) \longrightarrow$ limit $(\in \mathbb{R})$

2. $a_n \xrightarrow[\text{dim top}]{}$ limit $(\in \widehat{\mathcal{M}_k})$

Problem:

$\#_{\mathbb{F}_q}$ is not continuous w.r.t. dimensional topology.

(ex: $q^{2n}\mathbb{L}^{-n} \xrightarrow[\text{dim top}]{} 0$ but $\#_{\mathbb{F}_q}(q^{2n}\mathbb{L}^{-n}) \to \infty$)

Bertini type Theorems:

$X \subset \mathbb{P}^n$ smooth proj variety

$$\underset{\shortparallel}{U_d} \subset \Gamma(\mathbb{P}^n, \mathcal{O}(d))$$

open subset of hypersurfaces intersecting X transversely.

Thm: (Poonen '04) $k = \mathbb{F}_q$

$$\frac{|U_d(\mathbb{F}_q)|}{|\Gamma(\mathbb{P}^n, \mathcal{O}(d))(\mathbb{F}_q)|} \xrightarrow[d\to\infty]{} Z_X(q^{-\dim X - 1})^{-1}$$

$= |U_d(\mathbb{F}_q)|\, q^{-\dim \Gamma(\mathbb{P}^n, \mathcal{O}(d))}$

Thm: (Vakil - Wood '15) In $\widehat{\mathcal{M}_k}$

$$\frac{[U_d]}{[\Gamma(\mathbb{P}^n, \mathcal{O}(d))]} \xrightarrow[d\to\infty]{} Z_X^{Kap}(\mathbb{L}^{-\dim X - 1})^{-1}$$

$= [U_d]\, \mathbb{L}^{-\dim \Gamma(\mathbb{P}^n, \mathcal{O}(d))}$

$$Z_X^{Kap}(t) = \sum_{n \geq 1} [\underset{\substack{\shortparallel \\ X^n/S_n}}{\mathrm{Sym}^n X}]\, t^n$$

Question: how to compare these results when $k = \mathbb{F}_q$?

From now on: k finite, $= \mathbb{F}_q$.

Slogan: pass to zeta functions, i.e apply the <u>zeta measure</u>

$$
\begin{aligned}
\mathrm{zeta}: \mathcal{M}_k &\to R_1 = \{ f \in \mathbb{C}(t),\ f(0) = 1 \} \\
X &\mapsto Z_X(t) \\
\mathbb{L} &\mapsto \frac{1}{1-qt} \\
\mathbb{C}^n &\mapsto \frac{1}{1-q^n t}
\end{aligned}
$$

Ring structure:

$$R_1 \longrightarrow \mathbb{Z}[\mathbb{C}^\times]$$

$$f = \prod_a (1-at)^{-k_a} \longmapsto \sum_a k_a [a] \qquad (-\operatorname{div} f(t^{-1}))$$

$k_a \in \mathbb{Z}$

$$\frac{1}{1-qt} \mapsto [q]$$

Topologies on $\mathcal{R}_1$:

- coefficient topology:

$$\mathcal{R}_1 \xrightarrow[\text{exp at } 0]{\text{power series}} 1 + t\,\mathbb{C}[[t]]$$

$\parallel$

$\mathbb{C}^{\mathbb{N}}$ with product top

- weight topology:

$$\Big\| \sum_a k_a [a] \Big\|_\infty = \sup_{k_a \neq 0} |a|$$

- Hadamard topology:

$$\Big\| \sum_a k_a [a] \Big\|_H = \sum_a |k_a|\,|a|$$

Prop: the completion of $\mathcal{R}_1$ for $\|\cdot\|_H$ is

$$\left\{ \sum_a k_a[a] \ \text{(discretely supported)} \ \text{s.t.} \ \sum_a |k_a|\,|a| < \infty \right\}$$

$\sum k_a[a]$

$\downarrow\wr$

Hadamard factorisation

$$\mathcal{H}_1 = \left\{ \frac{f}{g},\ f, g \text{ entire functions of genus zero},\ f(0) = g(0) = 1 \right\}$$

$$\frac{\prod_{k_a<0} (1-at)^{-k_a}}{\prod_{k_a>0} (1-at)^{k_a}}$$

ring of Hadamard functions.

Principle: (Meta-conjecture) if

$$a_n \in \mathcal{M}_{\mathbb{F}_q}$$

is a "natural" sequence of classes s.t.

$$Z_{a_n}(t) \to f \in \mathcal{H}_1$$

both in the coeff and in the weight top, then it converges also in the Hadamard top.

Back to Bertini:

Conjecture: $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ smooth proj variety

$$U_d \subset \Gamma(\mathbb{P}^n, \mathcal{O}(d))$$

" hypersurfaces intersecting X transversely.

$$Z_{U_d}\left(q^{-\dim \Gamma(\mathbb{P}^n, \mathcal{O}(d))} t\right) \xrightarrow[\text{Hadamard top}]{} \underbrace{Z^{Kap}_{X, zeta}\left(\mathbb{L}_{zeta}^{-\dim X - 1}\right)^{-1}}_{\text{Hadamard function } \in \mathcal{H}_1}$$

Pattern-avoiding zero-cycles

Motivation from number theory

$a_1, \dots, a_m$ are said to be <u>relatively n-prime</u> if there does not exist $b \geq 2$ s.t.

$$\forall i \qquad b^n \mid a_i$$

$(\Leftrightarrow \gcd(a_1, \dots, a_m)$ n^{th}-power-free$)$

Thm: Given $m, n \geq 1$

$$\lim_{d \to \infty} \frac{\#\{(a_1, \dots, a_m) \in \{1, \dots, d\}^m,\ a_1, \dots, a_m \text{ rel. } n\text{-prime}\}}{d^m}$$

exists, and equals $\zeta(mn)^{-1}$.

X quasi-proj / k

Def: $X_n^{(d_1,\dots,d_m)} = \left\{ \begin{array}{l} (C_1,\dots,C_m) \in \mathrm{Sym}^{d_1}X \times \dots \times \mathrm{Sym}^{d_m}X \\ \text{s.t. } \forall x \in X, \text{ multiplicities of } x \\ \quad \text{in } C_1,\dots,C_m \text{ are} \\ \quad \underline{\text{not}} \text{ all } \geq n \end{array} \right\}$

Ex: $X_2^d = \mathrm{Conf}^d X \subset \mathrm{Sym}^d X$

Thm:

$$\lim_{d_1,\dots,d_m \to \infty} \frac{\left[X_n^{(d_1,\dots,d_m)}\right]_{zeta}}{\left[\mathrm{Sym}^{d_1}X \times \dots \times \mathrm{Sym}^{d_m}X\right]_{zeta}} \underset{\substack{\uparrow \\ \text{Hadamard} \\ \text{top}}}{=} Z^{Kap}_{X,zeta}\left(\mathbb{L}_{zeta}^{-mn\dim X}\right)^{-1}$$

Remarks:

- More generally holds for motivic measure
$$\varphi : \mathcal{M}_k \to R \qquad \text{(normed rg)}$$
satifying some natural conditions.

- lifts/generalizes results by

Farb - Wolfson - Wood 2019

Vakil - Wood 2015

- Proof: generating function argument

$$\frac{\sum_{\underline{d}} [X_n^{\underline{d}}]\, \underline{t}^{\underline{d}}}{\sum_{\underline{d}} [\mathrm{Sym}^{\underline{d}} X]\, \underline{t}^{\underline{d}}} = Z_X^{Kap}\left((t_1 \cdots t_m)^n \right)^{-1}$$

- more generally:

multiplicities $\not\ni v \quad \forall v \in V$

$(\not\ni (n, \dots, n))$